Intro GGT — Exercises 3

Exercise 1 (Some group theory preliminary)**.** Let H, H' be finite index subgroups of a group G, and let $N \triangleleft H$, $N' \triangleleft H'$ be finite normal subgroups.

1. Show that $(H \cap N')(H' \cap N)$ is a finite normal subgroup of $H \cap H'$.
2. Show that $(H \cap H')N/N$ is isomorphic to $(H \cap H')/(N \cap H')$.
   (Hint: the third isomorphism theorem.)
3. Show that $(H \cap H')N/N$ is a finite index subgroup of H/N.
4. Show that if $U_1, \dots, U_n$ is a sequence of groups such that U_{i+1} is a finite quotient of a finite index subgroup of U_i, then U_n is a finite quotient of a finite index subgroup of U_1.
   (Hint: Let H_1, H_2 be finite index subgroups of U_1, U_2 and look at the the epimorphisms
   $$\pi_1 : H_1 \twoheadrightarrow U_2 \qquad \pi_2 : H_2 \twoheadrightarrow U_3;$$
   consider $\pi_1^{-1}(H_2) \twoheadrightarrow H_2 \twoheadrightarrow U_3$.)
5. Deduce that $(H \cap H')/(H \cap N')(H' \cap N)$ is a finite quotient of a finite index subgroup of H.

Exercise 2 ((Weak) Commensurability)**.** Two groups G_1, G_2 are said to be **commensurable** if there exist finite index subgroups $H_1 < G_1, H_2 < G_2$ such that H_1 and H_2 are isomorphic.

1. Let H, H' be finite index subgroups of a group G. Show that $H \cap H'$ is a finite index subgroup of G.
2. Show that being commensurable is an equivalence relation.
3. Show that if G_1, G_2 are commensurable, then they are quasi-isometric.

Two groups G_1, G_2 are said to be **weakly commensurable** if there exist *finite index subgroups* $H_1 < G_1, H_2 < G_2$ and *finite normal subgroups* $N_1 \triangleleft H_1$, $N_2 \triangleleft H_2$ such that $H_1/N_1 \cong H_2/N_2$.

4. Using the results of Exercise 1, show that being weakly commensurable is an equivalence relation.
5. Show that if N is a finite subgroup of a group H, then H and H/N are quasi-isometric.
6. Deduce that if G_1, G_2 are weakly commensurable, then they are quasi-isometric.

Exercise 3 (Growth of the Heisenberg group)**.** Let

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y, z \in \mathbb{Z} \right\}.$$

1. Show that H with the matrix multiplication is a nilpotent group of class 2.

   This group is called the **Heisenberg group**.

2. Let
$$a = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \qquad c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Show that $S = \{a, b, c\}$ generates H.
   (Hint: Compute $b^y a^x c^z$.)

   (b) Show that $c = [a, b]$ and that c lies in the center of H. Deduce that H is 2-generated.

3. We still consider $S = \{a, b, c\}$ as the generating set for convenience.

   (a) Show that $c^{xy} = [a^x, b^y]$ for any $x, y \in \mathbb{Z}$.

   (b) Deduce that $d_S(1, c^{n^2}) \leq 4n$ for any $n \in \mathbb{N}$; and that $d_S(1, c^z) \leq 6\sqrt{|z|}$ for any $z \in \mathbb{Z}$.
   (Hint: Consider the unique $n \in \mathbb{N}$ such that $n^2 \leq |z| < (n+1)^2$.)

   (c) Deduce that there exists $A > 0$ such that $\beta_{H,S}(n) \geq An^4$.
   (Hint: Compute that number of $a^x b^y c^z \in H$ with $|x| \leq \frac{n}{5}$, $|y| \leq \frac{n}{5}$ and $|z| \leq \frac{n^2}{100}$.)

4. Let $h_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Denote $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (a) Compute $h_{x,y,z}a^{\pm}$, $h_{x,y,z}b^{\pm}$, $h_{x,y,z}c^{\pm}$.

   (b) Show that $d_S(I_3, h_{x,y,z}) \geq |x| + |y|$.

   (c) Show that $d_S(I_3, h_{x,y,z}) \geq \sqrt{|z|}$.

   (d) Deduce that $d_S(I_3, h_{x,y,z}) \geq \frac{1}{2}\left(|x| + |y| + \sqrt{|z|}\right)$.

   (e) Deduce that there exists $B > 0$ such that
$$\beta_{H,S}(n) \leq Bn^4.$$

5. Conclude that the Heisenberg group has polynomial growth of degree 4.